MATH 579 Exam 6 Solutions

1. (5-8 points) Let $\pi = (2\ 4)\ (1\ 5\ 3)$. Calculate and simplify $\pi \circ \pi \circ \pi$.

   $\pi \circ \pi \circ \pi = (2\ 4)\ (2\ 4)\ (2\ 4)\ (1\ 5\ 3)\ (1\ 5\ 3)\ (1\ 5\ 3) = (2\ 4)$.

2. Prove that $p(n)$ is equal to the number of partitions of $2n$ with no odd parts.

   We create a function f between partitions of n and partitions of $2n$, that acts by doubling each part. The range of this function consists of exactly those partitions of $2n$ with all even parts, and on that range f is invertible, hence it is a bijection.

3. Consider all partitions of 11. What is the maximal Durfee square? Give all partitions that yield this Durfee square.

   We can fit a 3×3 Durfee square into the partitions, but not a 4×4, since $9 < 11 < 16$. The remaining two boxes in the Ferrers diagram can both be to the right of the square $(5+3+3,\ 4+4+3)$, or one can be to the right and one below $(4+3+3+1)$, or they can both be below the square $(3+3+3+2,\ 3+3+3+1+1)$. Together, these are five partitions.

4. Consider $\pi \in S_n$. Prove that, for any such choice of π, that $|\det A| = 1$, for matrix A whose entries are given by $A_{i,j} = \begin{cases} 1 & \pi(i) = j \\ 0 & \text{otherwise.} \end{cases}$

   Note that A is a 0/1 matrix with exactly one 1 in each row and column. We will prove that all such matrices have determinant 1 or -1, by induction on the size n. If $n = 1$, then $A = [1]$, which has determinant 1. Otherwise, we expand A on the first row, which has just one nonzero entry, so $\det A = 1(\pm 1) \det B$, where B is a matrix obtained by deleting the first row and some column of A. Since B is of the same form, $|\det B| = 1$ by the inductive hypothesis, so $|\det A| = 1$.

5. (5-12 points) Prove that $\left[{n \atop 2} \right] \geq (n-1)! + (n-2)!$, for all integer $n \geq 2$.

   We want to count permutations of n with exactly two cycles. This includes those that have one cycle of length 1 and one cycle of length $n-1$ (and lots of others). These are counted by Thm 6.9, as $\frac{n!}{1!1!1^1(n-1)^1} = n \cdot (n-2)! = (n-1) \cdot (n-2)! + 1 \cdot (n-2)! = (n-1)! + (n-2)!$.

   In fact, $\left[{n \atop 2} \right] = (n-1)!(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1})$.